MA 681, Spring 2016

Assignment 6.

Complex integral. Cauchy Theorem

This assignment is due Wednesday, March 2. Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

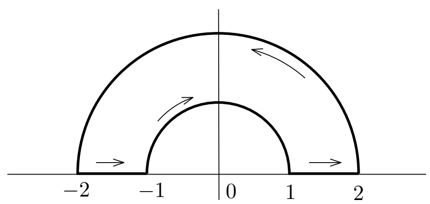
1. Integral

(1) Evaluate the integral

$$\int_L \frac{z}{\bar{z}} dz,$$

where L is a closed contour that bounds "upper semi-ring" $1 \leq |z| \leq 2$, $\operatorname{Im} z \geq 0$, traversed counterclockwise (see figure). (*Hint:* The answer is $4/3$).

Why this integral being nonzero does not contradict Cauchy Theorem?



(2) Evaluate the integral

$$\int_{|z-a|=R} (z-a)^n dz$$

($R > 0$) for all values of the integer n by parameterizing the path of integration and performing a direct computation.

(3) Prove that if $f(z)$ is continuous in the closed domain $|z| \geq R_0$, $0 \leq \arg z \leq \alpha$ ($0 \leq \alpha \leq 2\pi$), and if the limit

$$\lim_{z\to\infty} z f(z) = A$$

exists, then

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z) dz = iA\alpha,$$

where Γ_R is the arc of the circle $|z| = R$ lying in the given domain.

(*Hint:* Parametrize the arc by Re^{it}. Similarly to Problem 4 of HW5, write $f(z) = \frac{A+(zf(z)-A)}{z}$ in the integral.)

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2. Cauchy Theorem

Even if you remember Cauchy Integral Formula and residue calculus, don't use them in the problems below for fear of breaking lecture-time continuum and collapsing the universe. Rather, stick to Cauchy Theorem and Cauchy Theorem for multiple contours.

(4) Suppose that $f(z)$ is analytic at every point of the closed domain $0 \leq \arg z \leq \alpha$ $(0 \leq \alpha \leq 2\pi)$, and that $\lim_{z\to\infty} zf(z) = 0$. Prove that if the integral

$$J_1 = \int_0^\infty f(x)dx$$

exists, then so does the integral

$$J_2 = \int_L f(z)dz,$$

where L is the ray $z = re^{i\alpha}$, $0 \leq r \leq \infty$. Moreover, show that $J_1 = J_2$.

(*Hint:* Use Cauchy Theorem and the previous problem.)

(5) Prove that

$$\int_L \frac{dz}{z^2+1} = 0$$

if L is any closed rectifiable simple curve in the outside of closed unit disc, i.e. L is contained in the region $|z| > 1$.

Show that the equality is in general false for arbitrary closed rectifiable simple curves that miss zeros of $z^2 + 1$.

(6) Evaluate the integral

$$\int_{|z-i|=R} \frac{z^4+z^2+1}{z(z^2+1)}dz$$

as a function of $R > 0$. You may omit values of R for which the denominator turns to 0. (Hint: $\frac{z^4+z^2+1}{z(z^2+1)} = z + \frac{1}{z} - \frac{1}{2}(\frac{1}{z+i} + \frac{1}{z-i})$.)

(7) Fix a polynomial $p(z) = (z - z_1)(z - z_2)\cdots(z - z_n)$, $z_k \neq z_j$ when $k \neq j$. Let L be a simple closed rectifiable curve that does not pass through any of the points $z_1, \ldots, z_n$. How many distinct values of $\int_L \frac{dz}{p(z)}$ can one obtain, at most, by changing L? (In other words, the above integral is a function $J(L)$ of L. How many values can it take, at most?)